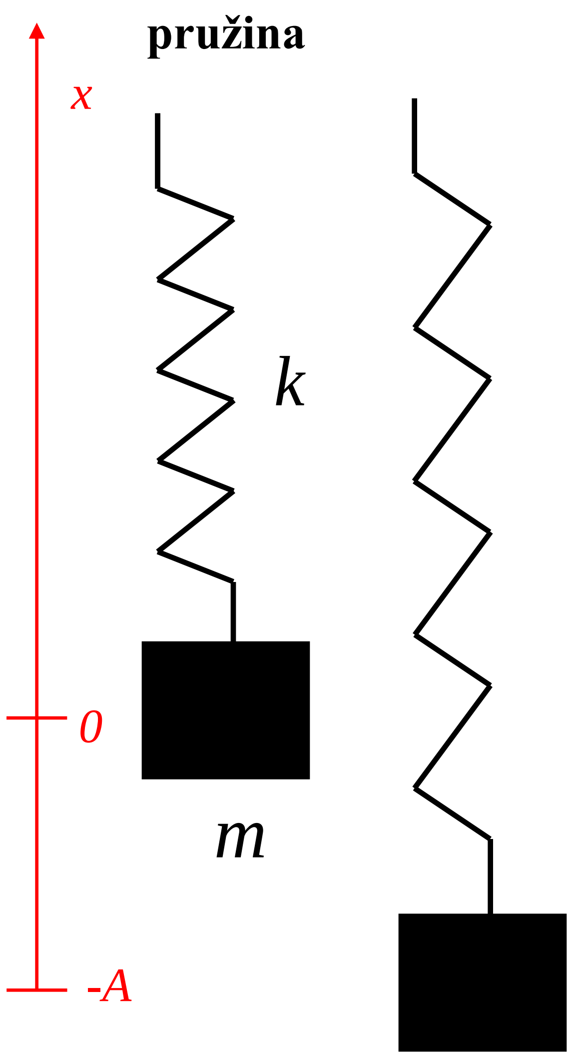
Harmonický oscilátor – pružina



$$a_x = \frac{F_x}{m}$$

$$F_x = -kx$$

$$a_x = -\frac{k}{m}x$$

$$\ddot{x} = -\frac{k}{m}x$$ **pohybová rovnice**

obecné řešení:

$$x(t) = A\sin(\omega t + \varphi)$$

úhlová frekvence $\omega = \sqrt{\frac{k}{m}}$

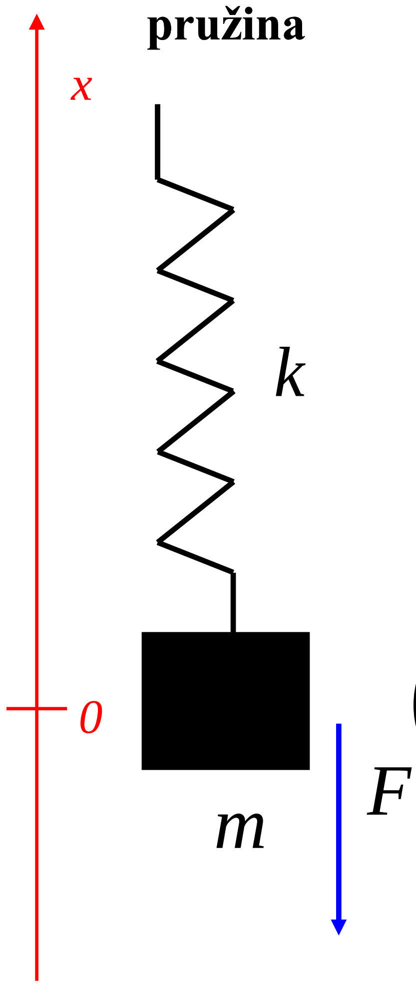
fázový posuv

$$x(t) = A\cos\varphi \sin\omega t + A\sin\varphi\cos\omega t$$

$$C_1 = A\cos\varphi, \quad C_2 = A\sin\varphi$$

$$x(t) = C_1 \sin\omega t + C_2\cos\omega t$$

Nucené kmity s tlumením



• budící síla:

$$F = F_0 \sin(\Omega t)$$

$$\omega_0^2 \equiv \frac{k}{m}, \quad 2\delta \equiv \frac{h}{m}$$

pohybová rovnice:

$$\ddot{x} + \frac{h}{m}\dot{x} + \frac{k}{m}x = \frac{F_0}{m}\sin(\Omega t)$$

$$\ddot{x} + 2\delta + \omega_0^2 x = \frac{F_0}{m}\sin(\Omega t)$$

• obecné řešení: $x(t) = C_1 e^{\alpha_1 t} + C_1 e^{\alpha_2 t} + x_p$

• partikulární řešení: $x_p = A_p \sin(\Omega t + \alpha)$

partikulární řešení:

$$\left(A_p\omega_0^2 - A_p\Omega^2\right)\sin(\Omega t + \alpha) + 2\delta A_p\Omega\cos(\Omega t + \alpha) = \frac{F_0}{m}\sin(\Omega t)$$

$$A_p\left[\left(\omega_0^2 - \Omega^2\right)\cos\alpha - 2\delta\Omega\sin\alpha\right]\sin(\Omega t) +$$

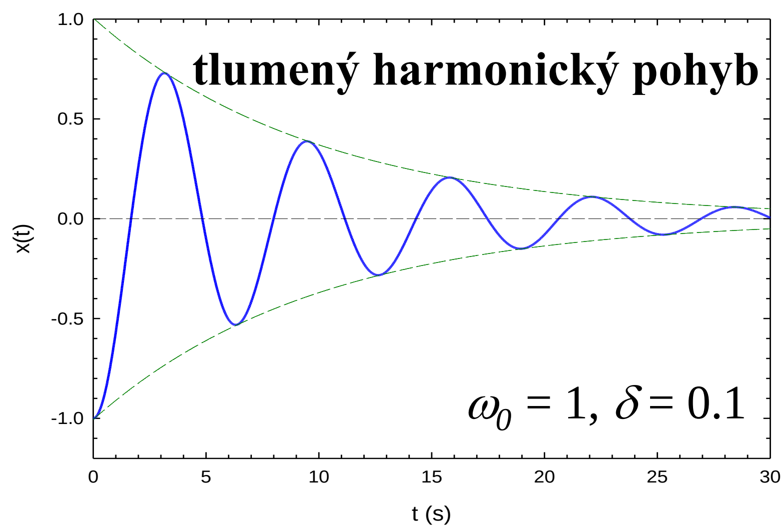
$$+ A_p\left[\left(\omega_0^2 - \Omega^2\right)\sin\alpha + 2\delta\Omega\cos\alpha\right]\cos(\Omega t) = \frac{F_0}{m}\sin(\Omega t)$$

Nucené kmity s tlumením – řešení v komplexní reprezentaci

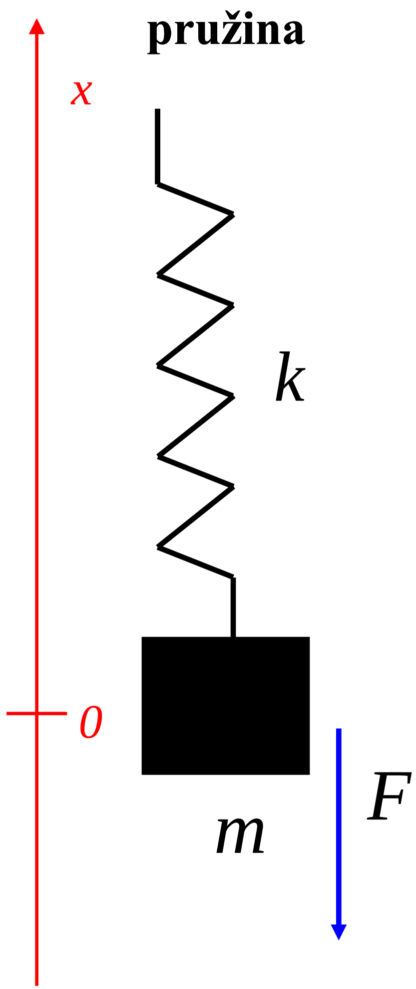
harmonický kmit: $x = A\sin(\omega t + \varphi) \longrightarrow Ae^{i(\omega t+\varphi)}$

A – amplituda, ω – úhlová frekvence, φ – fázový posuv

$$Ae^{i(\omega t+\varphi)} = A\cos(\omega t + \varphi) + iA\sin(\omega t + \varphi)$$



Nucené kmity s tlumením – řešení v komplexní reprezentaci



• budící síla:

$$F = F_0 e^{i\Omega t}$$

$$\omega_0^2 \equiv \frac{k}{m}, \quad 2\delta \equiv \frac{h}{m}$$

pohybová rovnice:

$$\ddot{x} + \frac{h}{m}\dot{x} + \frac{k}{m}x = \frac{F_0}{m}e^{i\Omega t}$$

$$\ddot{x} + 2\delta + \omega_0^2 x = \frac{F_0}{m}e^{i\Omega t}$$

• obecné řešení: $x(t) = C_1 e^{\alpha_1 t} + C_1 e^{\alpha_2 t} + x_p$

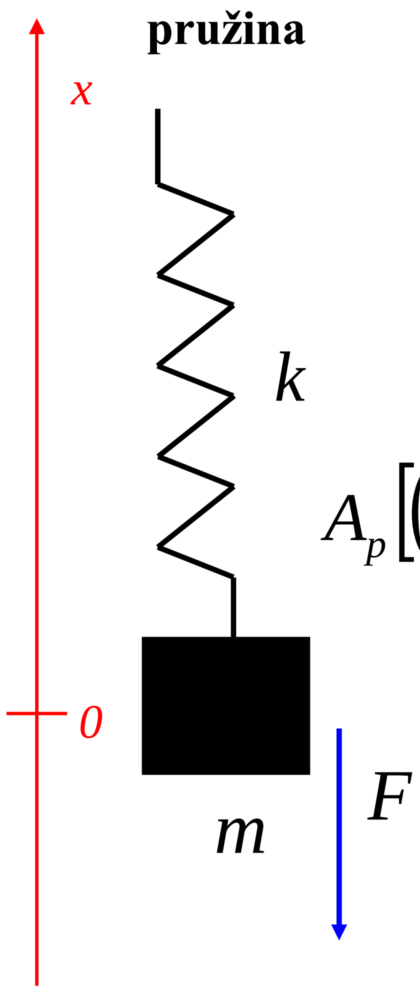
• partikulární řešení: $x_p = A_p e^{i(\Omega t + \alpha)}$

partikulární řešení:

$$\left(A_p\omega_0^2 - A_p\Omega^2\right)e^{i(\Omega t+\alpha)} + i2\delta A_p\Omega e^{i(\Omega t+\alpha)} = \frac{F_0}{m}e^{i\Omega t}$$

$$A_p\left[\left(\omega_0^2 - \Omega^2\right) + i2\delta\Omega\right] = \frac{F_0}{m}e^{-i\alpha}$$

Nucené kmity s tlumením – řešení v komplexní reprezentaci



• budící síla:

$$F = F_0 e^{i\Omega t}$$

pohybová rovnice:

$$\ddot{x} + \frac{h}{m}\dot{x} + \frac{k}{m}x = \frac{F_0}{m}e^{i\Omega t}$$

$$\omega_0^2 \equiv \frac{k}{m}, \quad 2\delta \equiv \frac{h}{m}$$

$$\ddot{x} + 2\delta + \omega_0^2 x = \frac{F_0}{m}e^{i\Omega t}$$

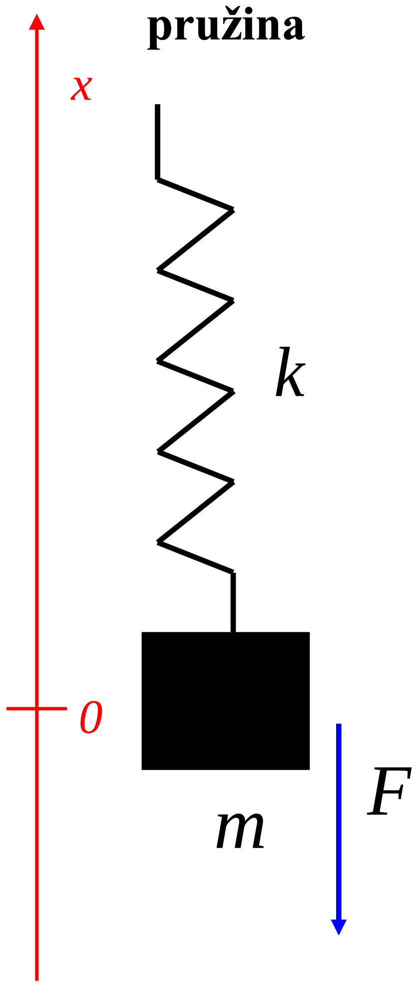
$$A_p\left[\left(\omega_0^2 - \Omega^2\right) + i2\delta\Omega\right] = \frac{F_0}{m}e^{-i\alpha} \quad \hat{K} = \left(\omega_0^2 - \Omega^2\right) + i2\delta\Omega = Ke^{i\beta}$$

$$K = \left|\hat{K}\right| = \sqrt{\left(\omega_0^2 - \Omega^2\right)^2 + 4\delta^2\Omega^2}, \quad \mathrm{tg}\beta = \frac{2\delta\Omega}{\omega_0^2 - \Omega^2}$$

$$A_p K e^{i\beta} = \frac{F_0}{m}e^{-i\alpha} \;\Rightarrow\; \mathrm{tg}\alpha = -\mathrm{tg}\beta = -\frac{2\delta\Omega}{\omega_0^2 - \Omega^2}$$

$$A_p = \frac{F_0}{mK} = \frac{F_0}{m\sqrt{\left(\omega_0^2 - \Omega^2\right)^2 + 4\delta^2\Omega^2}}$$

Nucené kmity s tlumením



• budící síla:

$$F = F_0 \sin(\Omega t)$$

$$\omega_0^2 \equiv \frac{k}{m}, \quad 2\delta \equiv \frac{h}{m}$$

pohybová rovnice:

$$\ddot{x} + \frac{h}{m}\dot{x} + \frac{k}{m}x = \frac{F_0}{m}\sin(\Omega t)$$

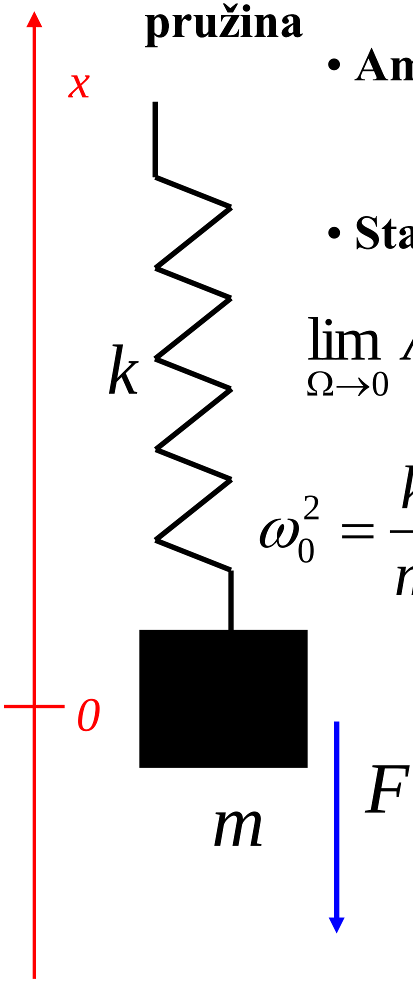
$$\ddot{x} + 2\delta + \omega_0^2 x = \frac{F_0}{m}\sin(\Omega t)$$

• **Partikulární řešení**:

$$x_p = \frac{F_0}{m\sqrt{\left(\omega_0^2 - \Omega^2\right)^2 + 4\delta^2\Omega^2}} \sin\left(\Omega t + \operatorname{arctg}\frac{2\delta\Omega}{\Omega^2 - \omega_0^2}\right)$$

• Část řešení: $C_1 e^{\alpha_1 t} + C_2 e^{\alpha_2 t}, \quad \alpha_{1,2} = -\delta \pm \sqrt{\delta^2 - \omega_0^2}$
konverguje k nule s rostoucím časem. Po dostatečně dlouhé době v **ustáleném stavu** lze tento člen zanedbat a za řešení pokládat pouze x_p.

Nucené kmity s tlumením v ustáleném stavu



- **Amplituda kmitů:**

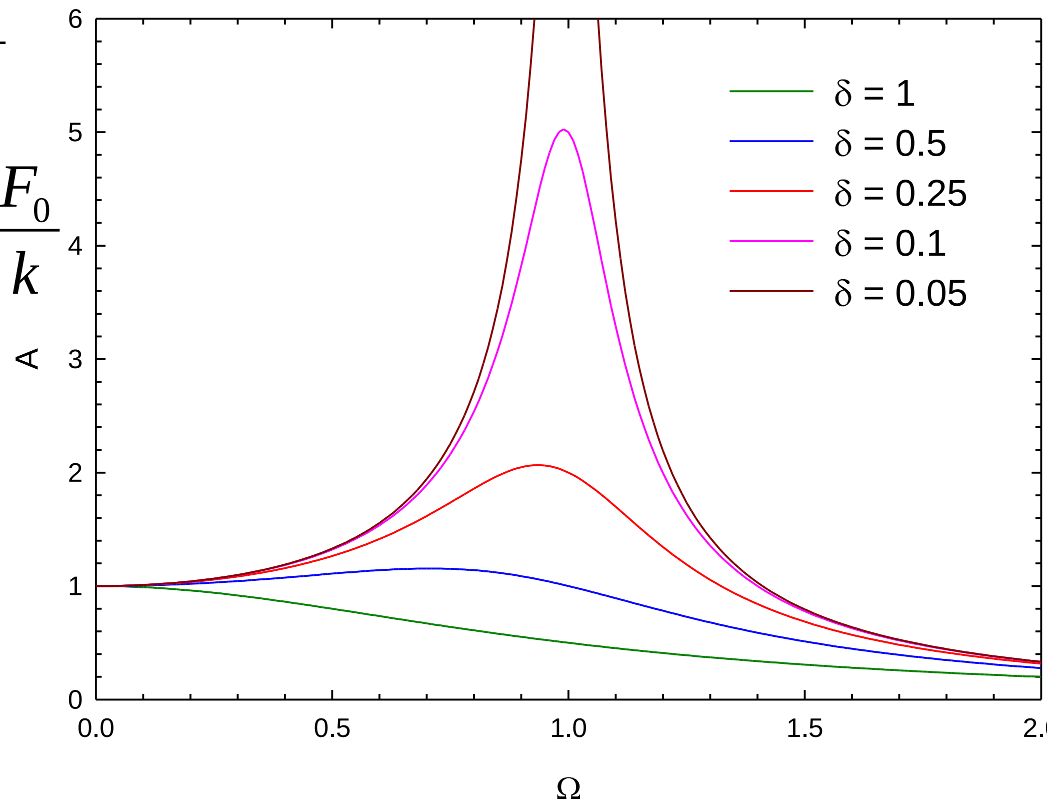
$$A_p = \frac{F_0}{m\sqrt{\left(\omega_0^2 - \Omega^2\right)^2 + 4\delta^2\Omega^2}}$$

- **Statická výchylka:**

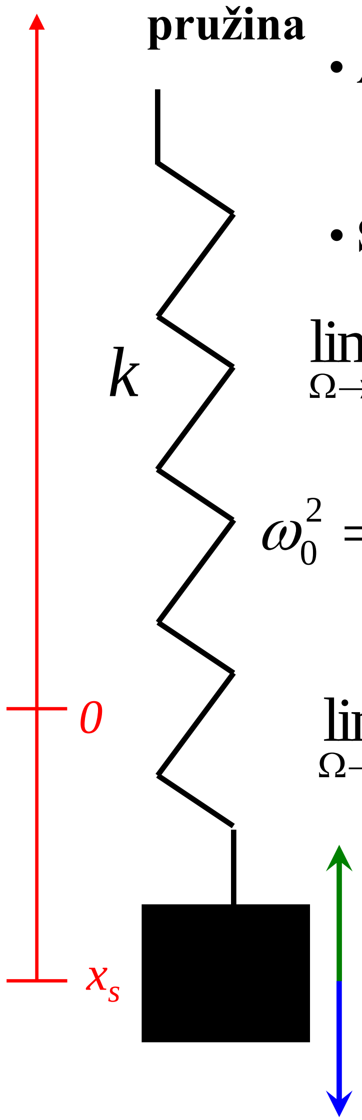
$$\lim_{\Omega \to 0} A_p = A_p(0) = \frac{F_0}{m\omega_0^2}$$

$$\omega_0^2 = \frac{k}{m} \Rightarrow x_s = \frac{F_0}{m\omega_0^2} = \frac{F_0}{k}$$

$\omega_0 = 1,\ F_0 = 1,\ m = 1$



Nucené kmity s tlumením v ustáleném stavu



- **Amplituda kmitů:**

$$A_p = \frac{F_0}{m\sqrt{\left(\omega_0^2 - \Omega^2\right)^2 + 4\delta^2\Omega^2}}$$

- **Statická výchylka:**

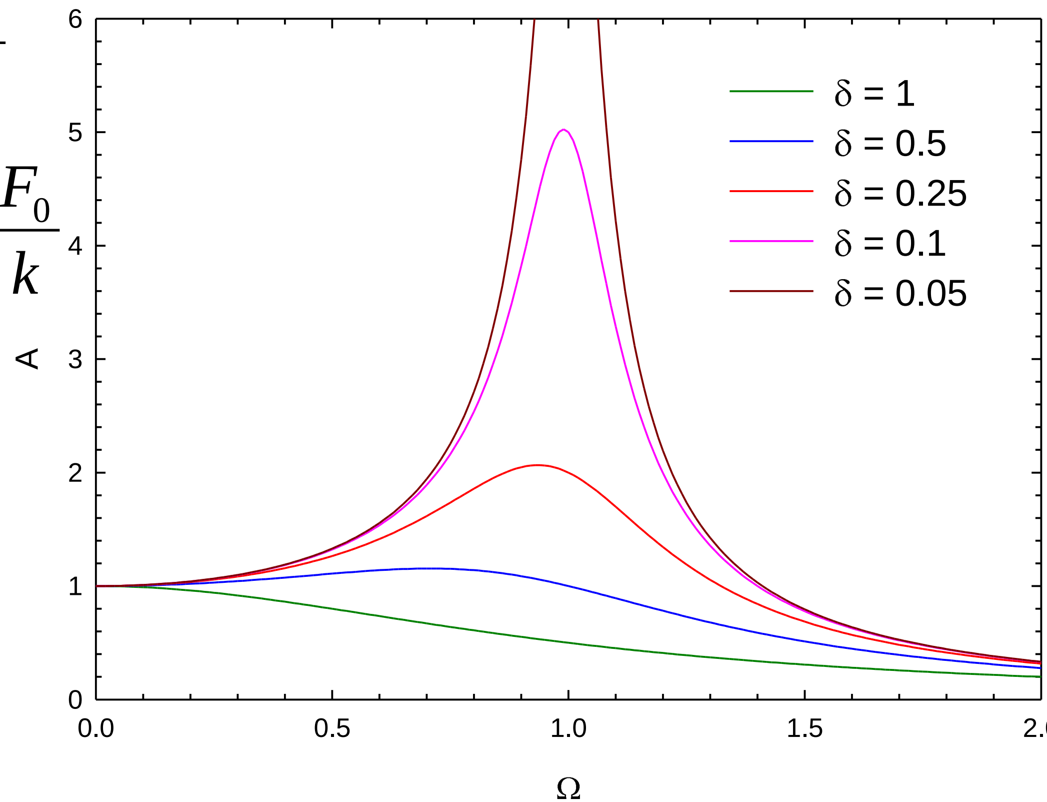
$$\lim_{\Omega\to 0} A_p = A_p(0) = \frac{F_0}{m\omega_0^2}$$

$$\omega_0^2 = \frac{k}{m} \Rightarrow x_s = \frac{F_0}{m\omega_0^2} = \frac{F_0}{k}$$

$$\lim_{\Omega\to\infty} A_p = A_p(\infty) = 0$$

$$F_x = kx$$

$\omega_0 = 1,\ F_0 = 1,\ m = 1$



Nucené kmity s tlumením v ustáleném stavu

• **Amplituda kmitů:**

$$A_p = \frac{F_0}{m\sqrt{\left(\omega_0^2 - \Omega^2\right)^2 + 4\delta^2\Omega^2}}$$

$$\frac{\mathrm{d}A_p}{\mathrm{d}\Omega} = \frac{2F_0\left(\omega_0^2 - \Omega^2 - 2\delta^2\right)\Omega}{m\left[\left(\omega_0^2 - \Omega^2\right)^2 + 4\delta^2\Omega^2\right]^{3/2}}$$

$$\frac{\mathrm{d}A_p}{\mathrm{d}\Omega} = 0$$

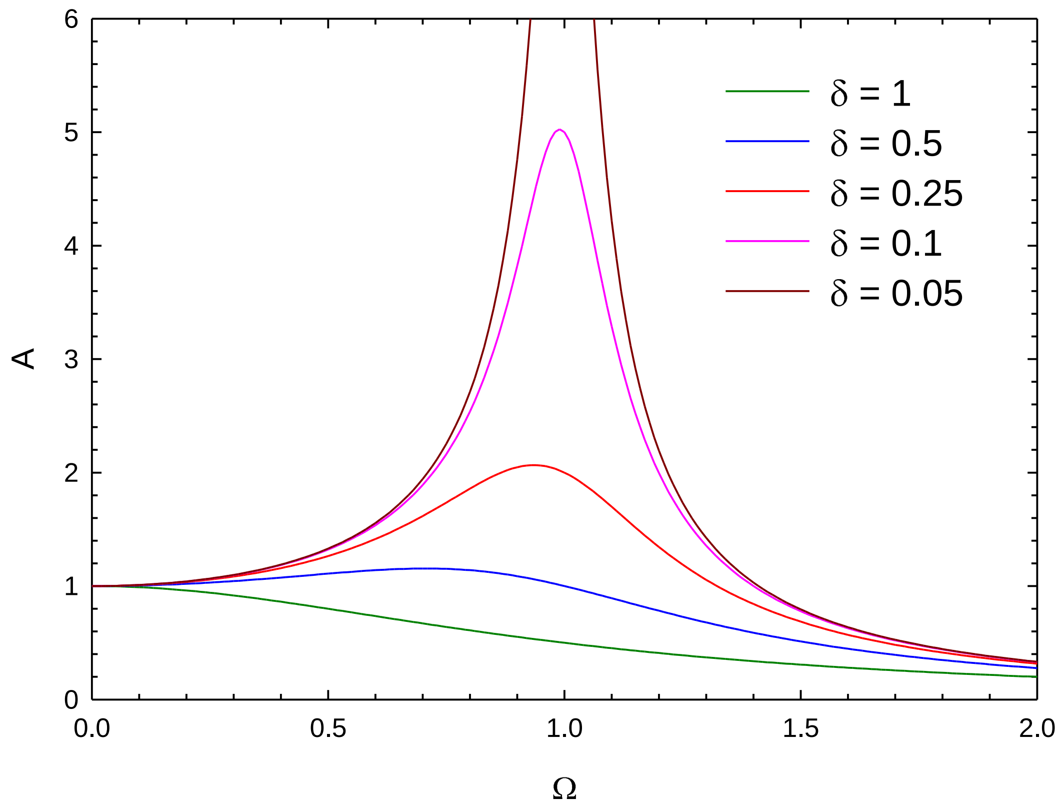
↓

• **rezonance amplitudy**

$$\Omega_r^2 = \omega_0^2 - 2\delta^2 \Rightarrow \Omega_r = \sqrt{\omega_0^2 - 2\delta^2}$$

$$A_p\left(\Omega_r\right) = \frac{F_0}{2m\delta\sqrt{\left(\omega_0^2 - \delta^2\right)}}$$

$\omega_0 = 1,\ F_0 = 1,\ m = 1$



Nucené kmity s tlumením v ustáleném stavu

- **Amplituda kmitů:**

$$A_p = \frac{F_0}{m\sqrt{\left(\omega_0^2 - \Omega^2\right)^2 + 4\delta^2\Omega^2}}$$

$$\frac{\mathrm{d}A_p}{\mathrm{d}\Omega} = \frac{2F_0\left(\omega_0^2 - \Omega^2 - 2\delta^2\right)\Omega}{m\left[\left(\omega_0^2 - \Omega^2\right)^2 + 4\delta^2\Omega^2\right]^{3/2}}$$

$$\frac{\mathrm{d}A_p}{\mathrm{d}\Omega} = 0$$

$\downarrow$

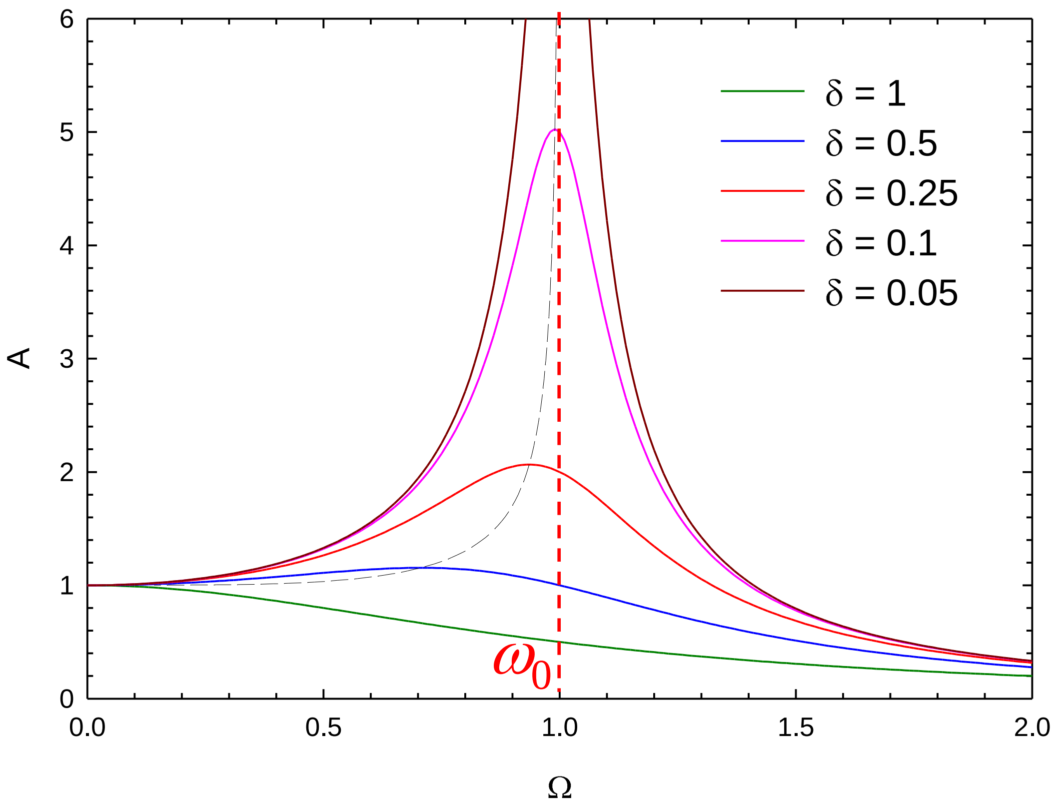
- **rezonance amplitudy**

$$\Omega_r^2 = \omega_0^2 - 2\delta^2 \Rightarrow \Omega_r = \sqrt{\omega_0^2 - 2\delta^2}$$

$$A_p\left(\Omega_r\right) = \frac{F_0}{2m\delta\sqrt{\left(\omega_0^2 - \delta^2\right)}}$$

$$\lim_{\delta \to 0} A_p\left(\Omega_r\right) = \infty \qquad x_s = A_p(0) = \frac{F_0}{m\omega_0^2}$$

$\omega_0 = 1,\ F_0 = 1,\ m = 1$



Mechanická rezonance

- Tacoma Narrows Bridge (1940), Tacoma, Washington U.S.